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Sets defining minimal vertex covers[1]

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Abstract

Let V be a finite non-empty set of n elements, let r be a natural integer and let $\mathscr{A}=\{A_1,\dots,A_r\} \subset \mathscr{P}(V)$ be a non-empty set of subsets of V. In this paper the question arises whether there exists a graph G with vertex set V and whose minimal vertex cover sets are the elements of $\mathscr{A}$. In order to solve this problem we characterize the set of minimal vertex covers of a graph by using special binary matrices.

Keywords: Minimal vertex cover set; Graphs and matrices

1. Introduction

Let $G=(V(G),E(G))$ be a finite simple graph (without loops), with vertex set $V(G)$ and with edge set $E(G)$. A vertex $u \in V(G)$ and an edge $e \in E(G)$ are said to cover each other in G if they are incident in G (that is, if there is a vertex $v \in V(G)$ such that $e=[u,v] \in E(G)$). A vertex cover set in G is a set of vertices that covers all edges of G. Let us denote by $\Gamma(G)$ the set whose elements are the minimal vertex cover sets of G.

In this paper we discuss the following problem concerning minimal vertex cover sets of a graph. Given $V=\{x_1,\dots,x_n\}$ a finite non-empty set of n points, and given $\mathscr{A}=\{A_1,\dots,A_r\} \subset \mathscr{P}(V)$ a non-empty family of r subsets of V, does there exist a graph G with vertex set $V(G)=V$ and such that $\Gamma(G)=\mathscr{A}$?

It seems clear that the solution of our problem depends on the relative position of the elements that define the sets $A_1,\dots,A_r$. So, in order to solve the problem, it leads us to consider the incidence matrix $A(\mathscr{A})$ associated to the family $\mathscr{A}$ and, now, we must look for properties of this binary matrix that help us see if the answer is yes or no.

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Following this idea we establish our main result (see Theorem 2.1): there exists a graph G with vertex set V and whose minimal vertex cover sets are the elements of $\mathscr{A}$ if, and only if, the matrix $A(\mathscr{A})$ associated to $\mathscr{A}$ satisfies conditions (P1)–(P3), (see Section 2 for the definition of these conditions). Furthermore, in such a case, we demonstrate that such a graph is unique, and we describe the edge set of this graph.

The reader is referred to any one of the following Refs. [1,2] for graph theoretical concepts used here.

2. A characterization result

Let $V=\{x_1,\dots,x_n\}$ be a finite non-empty set of n points, let $r\geqslant 1$ be a natural integer and let $\mathscr{A}=\{A_1,\dots,A_r\}\subset\mathscr{P}(V)$ be a family of r subsets of V. Then we denote by $A(\mathscr{A})$ the incidence matrix associated to the family $\mathscr{A}$. That is to say:

$A(\mathscr{A})=(a_{i,j})\in\mathscr{M}_{r,n}(\{0,1\})$ is the binary matrix whose elements $a_{i,j}$ are defined as $a_{i,j}=0$ if $x_j\notin A_i$, and $a_{i,j}=1$ if $x_j\in A_i$.

Furthermore, we will use the following conditions defined on an arbitrary binary matrix. Let $M=(m_{i,j})\in\mathscr{M}_{r,n}(\{0,1\})$ be a matrix with r rows and n columns and where its elements $m_{i,j}$ are either zero or one. In such a case we say that the matrix M satisfies the condition:

(P1) if, for any two different integers $1\leqslant i_1,i_2\leqslant r$, there exists an integer $j\in\{1,\dots,n\}$ such that $m_{i_1,j}>m_{i_2,j}$.
That is, for any two differents rows i_1 and i_2 of M there exists an integer j such that the jth component of the row i_1 is equal to one, while the jth component of the row i_2 is equal to zero.

(P2) if, for any integer $1\leqslant j\leqslant n$, there exists an integer $i\in\{1,\dots,r\}$ such that $m_{i,j}=0$.
That is, each column of the matrix M has a zero element.

(P3) if, for any $l\geqslant 2$ different integers $1\leqslant j_1,\dots,j_l\leqslant n$ such that $m_{i,j_1}+\cdots+m_{i,j_l}\geqslant 1$ for all $i\in\{1,\dots,r\}$, then there are two different integers $j_{\alpha_1},j_{\alpha_2}\in\{j_1,\dots,j_l\}$ such that $m_{i,j_{\alpha_1}}+m_{i,j_{\alpha_2}}\geqslant 1$ for all $i\in\{1,\dots,r\}$.
That is, if the vector sum of $l\geqslant 2$ different columns of the matrix M has all its components greater than or equal to one then, there are at least two of these l columns such that its vector sum already has all its components greater than or equal to one.

With the notations described above, the following theorem characterizes whenever the family $\mathscr{A}$ defines the minimal vertex cover sets of a graph:

Theorem 2.1. *Let $V=\{x_1,\dots,x_n\}$ be a finite non-empty set of n points. Let $r\geqslant 1$ be a natural integer and let $\mathscr{A}=\{A_1,\dots,A_r\}$ be a family of r subsets of V. Then, the* *following statements hold:*

(a) *There exists a graph G with vertex set* $V(G)=V$ *and with minimal vertex cover sets* $\Gamma(G)=\mathscr{A}$ *if, and only if, the matrix* $A(\mathscr{A})$ *associated to the family* $\mathscr{A}$ *satisfies conditions* (P1), (P2) *and* (P3).

(b) *If there exists such a graph G then it is unique and its edge set is* $E(G)=\{[x_\alpha,x_\beta]$ *such that the vector sum of the* α *and* β *columns of the matrix* $A(\mathscr{A})$ *has all its components greater than or equal to one*$\}$.

Before doing the proof of Theorem 2.1 let us show an example.

Example 2.2. Let $V=\{1,2,3,4\}$ and let us consider the families of subsets $\mathscr{A}_1=\{\{1\},\{2\},\{1,3,4\}\}$, $\mathscr{A}_2=\{\{1,2\},\{1,3,4\}\}$, $\mathscr{A}_3=\{\{1,4\},\{2,4\},\{3,4\},\{1,3\}\}$, and $\mathscr{A}_4=\{\{1,4\},\{2,4\},\{1,2,3\}\}$. Then, the matrix

$$A(\mathscr{A}_1)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&1\end{pmatrix}$$ satisfies conditions (P2) and (P3) but does not satisfy (P1);

while

$$A(\mathscr{A}_2)=\begin{pmatrix}1&1&0&0\\1&0&1&1\end{pmatrix}$$ satisfies conditions (P1) and (P3) but does not satisfy (P2);

whereas

$$A(\mathscr{A}_3)=\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\1&0&1&0\end{pmatrix}$$ satisfies conditions (P1) and (P2) but does not satisfy (P3); and

$$A(\mathscr{A}_4)=\begin{pmatrix}1&0&0&1\\0&1&0&1\\1&1&1&0\end{pmatrix}$$ satisfies conditions (P1)–(P3).

Therefore applying Theorem 2.1 it follows that, for $i=1,2,3$ there does not exist a graph G_i with vertex set $V(G_i)=V$ and whose minimal vertex cover sets are the elements of the family $\mathscr{A}_i$; whilst there exists a unique graph G_4 with vertex set $V(G_4)=V$ and with minimal vertex cover sets $\Gamma(G_4)=\mathscr{A}_4$. Furthermore, the edge set $E(G_4)$ of the graph G_4 is $E(G_4)=\{[1,2],[1,4],[2,4],[3,4]\}$.

Proof of Theorem 2.1. *Proof* (b). For a given graph G we claim that $[x_\alpha,x_\beta]\in E(G)$ if, and only if, either $x_\alpha\in C$ or $x_\beta\in C$ for all minimal vertex cover set C of G. Therefore, from our claim it follows that, if G_1 and G_2 are two graphs such that $V(G_1)=V(G_2)=V$ and such that $\Gamma(G_1)=\Gamma(G_2)=\mathscr{A}$ then $E(G_1)=E(G_2)=\{[x_\alpha,x_\beta]$ such that the vector sum of the α and β columns of the matrix $A(\mathscr{A})$ has all its components greater than or equal to one} and, hence, $G_1 = G_2$. So, in order to prove (b), we only need to show our claim.

Let G be a graph. Then, from the definition of vertex cover set it follows that if $[x_\alpha, x_\beta] \in E(G)$ then either $x_\alpha \in C$ or $x_\beta \in C$ for all minimal vertex cover set C of G. Conversely, assuming that either $x_\alpha \in C$ or $x_\beta \in C$ for all $C \in \Gamma(G)$, we demonstrate that $[x_\alpha, x_\beta] \in E(G)$. If $[x_\alpha, x_\beta] \notin E(G)$, then $\{x_\alpha, x_\beta\}$ is an independent set of vertices of G and, hence, it follows that there exists $W \subset V(G)$ a maximal independent set of vertices of G such that $\{x_\alpha, x_\beta\} \subset W$. Clearly, the set $V(G) - W$ is a minimal vertex cover set of G. Therefore $x_\alpha, x_\beta \notin V(G) - W \in \Gamma(G)$, which is a contradiction. Hence, our claim follows, and this completes the proof of (b).

Proof (a$\Rightarrow$). Assume that there exists a graph G with vertex set $V(G) = V$ and with minimal vertex cover sets $\Gamma(G) = \mathscr{A}$. We want to prove that then conditions (P1)–(P3) must hold.

It is clear that condition (P1) holds since if $a_{i_1,j} \leqslant a_{i_2,j}$ for all $1 \leqslant j \leqslant n$, then $A_{i_1} \subset A_{i_2}$, which is a contradiction since $A_{i_1}, A_{i_2} \in \mathscr{A} = \Gamma(G)$ are minimal sets.

Let us show (P2). Let $1 \leqslant j \leqslant n$. Assume that $a_{i,j} = 1$ for all $i \in \{1, \dots, r\}$. Let $W_i = V(G) - A_i$. Since $\{A_1, \dots, A_r\}$ are all the minimal vertex cover sets of G then it follows that, $\{W_1, \dots, W_r\}$ are all the maximal independent sets of vertices of G. Likewise, since $a_{i,j} = 1$ for all $i \in \{1, \dots, r\}$, then $x_j \in A_i$ and, thus, we have that $x_j \notin W_i$. Therefore $[x_\alpha, x_j] \in E(G)$ for all $\alpha \in \{1, \dots, n\} - \{j\}$ and, hence, $C_j = \{x_\alpha$ where $\alpha \in \{1, \dots, n\}$ with $\alpha \neq j\}$ is a minimal vertex cover set of G. Thus, there exists an integer $i_j \in \{1, \dots, r\}$ such that $A_{i_j} = C_j$. Therefore $x_j \notin A_{i_j}$ and, hence, $a_{i_j,j} = 0$, which is a contradiction. So, condition (P2) holds.

To finish the proof of this implication let us demonstrate that condition (P3) holds. Let $1 \leqslant j_1, \dots, j_l \leqslant n$ be $l \geqslant 2$ different integers such that $a_{i,j_1} + \cdots + a_{i,j_l} \geqslant 1$ for all $i \in \{1, \dots, r\}$. We want to show that then there are two different integers $j_{\alpha_1}, j_{\alpha_2} \in \{j_1, \dots, j_l\}$ such that $a_{i,j_{\alpha_1}} + a_{i,j_{\alpha_2}} \geqslant 1$ for all $i \in \{1, \dots, r\}$. Since $A_1, \dots, A_r$ are the minimal vertex cover sets of G then it follows that, $a_{i,j_{\alpha_1}} + a_{i,j_{\alpha_2}} \geqslant 1$ for all $i \in \{1, \dots, r\}$ if, and only if, either $x_{j_{\alpha_1}} \in A_i$ or $x_{j_{\alpha_2}} \in A_i$ for all $i \in \{1, \dots, r\}$ if, and only if, $[x_{j_{\alpha_1}}, x_{j_{\alpha_2}}] \in E(G)$. Therefore, we must prove that there are two different integers $j_{\alpha_1}, j_{\alpha_2} \in \{j_1, \dots, j_l\}$ such that $[x_{j_{\alpha_1}}, x_{j_{\alpha_2}}] \in E(G)$.

Assume that the vertices $\{x_{j_1}, \dots, x_{j_l}\}$ are pairwise independent. Then there exists a maximal independent set of vertices W of G such that $\{x_{j_1}, \dots, x_{j_l}\} \subset W$. Since W is a maximal independent set of vertices of G then it follows that, there exists $i_0 \in \{1, \dots, r\}$ such that $V(G) - W = A_{i_0}$. So we have that $x_{j_1}, \dots, x_{j_l} \notin A_{i_0}$ and, thus, $a_{i_0,j_1} + \cdots + a_{i_0,j_l} = 0$ which is a contradiction of our assumption. Therefore condition (P3) holds, as we wanted to prove. This completes the proof of the implication ($\Rightarrow$).

Proof (a $\Leftarrow$). Let $V = \{x_1, \dots, x_n\}$ be a finite non-empty set. Let $r \geqslant 1$ be a natural integer and let $\mathscr{A} = \{A_1, \dots, A_r\} \subset \mathscr{P}(V)$ be a finite non-empty family of subsets of V. Let $A(\mathscr{A})$ be the matrix associated to the family $\mathscr{A}$. Assume that $A(\mathscr{A})$ satisfies conditions (P1)–(P3). We want to demonstrate that then there is a graph G with vertex set $V(G) = V$ and with minimal vertex cover sets $\Gamma(G) = \mathscr{A}$.

Let us define the graph $G=(V(G),E(G))$ as $V(G)=V$ and $E(G)=\{[x_\alpha,x_\beta]$ such that $a_{i,\alpha}+a_{i,\beta}\geqslant 1$ for all $1\leqslant i\leqslant r\}$. (Notice that from condition (P2) it follows that if $\alpha\in\{1,\dots,n\}$ then there is an integer $1\leqslant i\leqslant r$ such that $a_{i,\alpha}=0$ and, so, $[x_\alpha,x_\alpha]\notin E(G)$). We only must show that $\Gamma(G)=\mathscr{A}$.

Firstly let us prove that $A_1,\dots,A_r$ are vertex cover sets of G. Let $[x_\alpha,x_\beta]\in E(G)$. Then, from the definition of $E(G)$ it follows that $a_{i,\alpha}+a_{i,\beta}\geqslant 1$ for all $1\leqslant i\leqslant r$. Therefore either $a_{i,\alpha}=1$ or $a_{i,\beta}=1$ and, hence, it follows that either $x_\alpha\in A_i$ or $x_\beta\in A_i$. So A_i is a vertex cover set of G.

In order to finish the proof we only need to show that if C is a vertex cover set then there exists an integer $i_c\in\{1,\dots,r\}$ such that $A_{i_c}\subset C$ since, then, $\Gamma(G)\subset\{A_1,\dots,A_r\}$ and, so, from condition (P1), it follows that $\Gamma(G)=\{A_1,\dots,A_r\}=\mathscr{A}$.

Let C be a vertex cover set of G. Assume that $A_i\not\subset C$ for all $i\in\{1,\dots,r\}$. Then, for all $i\in\{1,\dots,r\}$, pick an element $x_{\alpha_i}\in A_i$ such that $x_{\alpha_i}\notin C$. Now we can write the set $\{x_{\alpha_1},\dots,x_{\alpha_r}\}$ as $\{x_{\alpha_1},\dots,x_{\alpha_r}\}=\{x_{j_1},\dots,x_{j_l}\}$ where $l\leqslant r$ and where the elements $x_{j_1},\dots,x_{j_l}$ are all different.

We claim that $l\geqslant 2$. Otherwise, if $l=1$, then $\{x_{j_1}\}=\{x_{\alpha_1},\dots,x_{\alpha_r}\}$. Hence $x_{j_1}\in A_i$ for all $i\in\{1,\dots,r\}$. Thus, $a_{i,j_1}=1$ for all $i\in\{1,\dots,r\}$ which contradicts condition (P2). Therefore $l\geqslant 2$.

For all $i\in\{1,\dots,r\}$ we have that there exists an integer $j_{\alpha(i)}\in\{j_1,\dots,j_l\}$ such that $x_{\alpha_i}=x_{j_{\alpha(i)}}$. Therefore $x_{j_{\alpha(i)}}\in A_i$ and, hence, $a_{i,j_1}+\cdots+a_{i,j_l}\geqslant a_{i,j_{\alpha(i)}}=1$. Applying condition (P3) it follows that there are two different integers $j_{\beta_1},j_{\beta_2}\in\{j_1,\dots,j_l\}$ such that $a_{i,j_{\beta_1}}+a_{i,j_{\beta_2}}\geqslant 1$ for all $i\in\{1,\dots,r\}$. Hence, from the definition of G it follows that $[x_{j_{\beta_1}},x_{j_{\beta_2}}]\in E(G)$. Likewise, C is a vertex cover set of G and, thus, it follows that either $x_{j_{\beta_1}}\in C$ or $x_{j_{\beta_2}}\in C$, which contradicts our choice of the elements $\{x_{j_1},\dots,x_{j_l}\}$.

Therefore, any vertex cover set contains at least one of the elements of the family $\mathscr{A}$. This completes the proof of this implication and, hence, the proof of the theorem. □

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References

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